

ON THE DRINFELD TWIST FOR QUANTUM $sl(2)$

Ludwik Dąbrowski

SISSA, Via Beirut 2-4, Trieste, Italy.

E-MAIL: DABROW@SISSA.IT

Fabrizio Nesti

SISSA, Via Beirut 2-4, Trieste, Italy.

E-MAIL: NESTI@SISSA.IT

Pasquale Siniscalco

SISSA, Via Beirut 2-4, Trieste, Italy.

E-MAIL: SINIS@SISSA.IT

Abstract

An isomorphism, up to a twist, between the quasitriangular quantum enveloping algebra $U_h(sl(2))$ and the (classical) $U(sl(2))[[h]]$ is discussed. The universal twisting element \mathcal{F} is given up to the second order in the deformation parameter h .

1 Introduction

In 1989 Drinfeld showed by cohomological arguments that, as a formal series in a deformation parameter \hbar , all the quantum symmetries (quasitriangular Hopf algebras) $U_\hbar(g)$, where g is a semisimple Lie algebra, are isomorphic with $U(g)[[\hbar]]$, up to a twist \mathcal{F} [1, 2]. He also posed a problem (cf.[2]) to find a concrete pair (m, \mathcal{F}) , consisting of an isomorphism m and a universal twisting element \mathcal{F} . This turns out to be a formidable task, which as far as we know, is not yet solved in general. The only case when it has been performed concerns the q -deformed Heisenberg algebra $\mathcal{H}_q(1)$ [3]. The next important case to be investigated is the quantum deformation of $sl(2)$ (as a matter of fact $\mathcal{H}_q(1)$ can be obtained from it by a contraction). As far as $U_\hbar(sl(2))$ is regarded, a candidate for the isomorphism m is actually known [4]. Also, a series of related particular matrix solutions for the twist element \mathcal{F} were reported, namely \mathcal{F} in the representations $\frac{1}{2} \otimes j$, where $\frac{1}{2}$ denotes the fundamental representation and j denotes the irreducible $(2j+1)$ -dimensional representation of $sl(2)$ [5, 6], (see also [7]). Moreover, in [8] a sort of a ‘semi-universal’ form of \mathcal{F} has been given, i.e. the expression for $(\frac{1}{2} \otimes \text{id})(\mathcal{F})$. However, the universal element \mathcal{F} itself has not been known beyond the first order in the deformation parameter \hbar (the first order coefficient being given by the classical r -matrix r). In this letter, we investigate and report the solution up to the second order in \hbar . In the subsequent sections we separately discuss the problem on the levels of algebra, Hopf algebra and quasitriangular Hopf algebra.

It is worth to mention that evaluating \mathcal{F} in the representation $\rho_L \otimes \rho_L$, where ρ_L is the representation of $sl(2)$ in terms of the left-invariant vector fields on $SL(2)$, one obtains a quantization of the Lie-Poisson bracket on $SL(2)$ given by r [9]. In particular, the second coefficient of $(\rho_L \otimes \rho_L)(\mathcal{F})$ provides an interesting second order (bi)differential operator on $SL(2)$.

2 Algebra level

We start by specifying our conventions about Lie algebra $sl(2)$. The generators are H, E, F with the commutation relations:

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = H. \quad (2.1)$$

As a consequence we have the following exchange relations between any polynomial $\phi(H)$ in H and the powers of E and F :

$$\begin{aligned} \phi(H)E^n &= E^n\phi(H+n), \\ \phi(H)F^n &= F^n\phi(H-n). \end{aligned} \quad (2.2)$$

The quadratic Casimir element in the universal enveloping algebra $U(sl(2))$ is

$$I = 2EF + H(H - 1) = 2FE + H(H + 1) \doteq j(j + 1) . \quad (2.3)$$

A possible basis for the enveloping algebra is provided by the set $\{H^l E^m F^n\}$, but using the relations (2.3) we can pass to the basis given by $\{H^a I^b E^c \oplus H^r I^s F^t\}$. This basis will be more suitable for our computations.

Next, the generators J^+, J^-, J^0 of the q -deformed algebra obey the following commutation relations:

$$[J^0, J^+] = J^+, \quad [J^0, J^-] = -J^-, \quad [J^+, J^-] = \frac{1}{2}[2J^0] . \quad (2.4)$$

where $[x]$, the q -analogue of x , is defined as:

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} . \quad (2.5)$$

The ‘deforming maps’ introduced in [4], provide (cf.[10]) an isomorphism m between $U_h(sl(2))$ and $U(sl(2))[[h]]$ which is given by mapping the generators J^0, J^+, J^- to the following combinations of H, E, F

$$J^0 \rightarrow H, \quad J^+ \rightarrow \phi^+ E, \quad J^- \rightarrow \phi^- F = F \phi^+ , \quad (2.6)$$

where

$$\phi^\pm = \sqrt{\frac{[j \pm H][1 + j \mp H]}{(j \pm H)(1 + j \mp H)}} . \quad (2.7)$$

We remark that (2.7) is a well defined expression, as the inverse and square root operations are admissible in the h -adic topology. In fact, with $q = e^h$, we can write the expansion in h up to the second order as

$$\phi^\pm = 1 + \frac{1}{12}h^2 (2I + 2H(H \mp 1) - 1) + o(h^3) \doteq 1 + h^2 \phi_2^\pm + o(h^3) . \quad (2.8)$$

It will be useful to mention [11], that any other isomorphism m' can differ at most by a similarity via an invertible element $M \in U(sl(2))[[h]]$, i.e.

$$m' = M m M^{-1} . \quad (2.9)$$

We conclude this section with few remarks. Note that (2.6) is in fact valid also for the $*$ -algebras $U_h(su(2))$ and $U(su(2))[[h]]$, since it fulfills the relevant hermicity condition. In this respect, more general isomorphisms belonging to the one-parameter family introduced in [4] do not satisfy such a hermicity requirement. In addition, they are not suitable for our purposes since the coefficients of the expansion in h are not polynomial in the generators.

3 Hopf algebra level

The enveloping algebra $U(sl(2))[[h]]$ with relations (2.1) when equipped with the usual coproduct

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \forall x \in sl(2), \quad (3.10)$$

becomes a Hopf algebra. In the quantum case, the coproduct in $U_h(sl(2))$ is defined as:

$$\begin{aligned} \Delta_q(J^0) &= 1 \otimes J^0 + J^0 \otimes 1, \\ \Delta_q(J^\pm) &= J^\pm \otimes q^{J^0} + q^{-J^0} \otimes J^\pm. \end{aligned} \quad (3.11)$$

(The counit and coinverse are not needed for our purposes).

The main part of Drinfeld Theorem guarantees that these two classical and quantum coproducts are related via a twist by an invertible $\mathcal{F} \in (U(sl(2)) \otimes U(sl(2)))[[h]]$.

More precisely, defining

$$\widetilde{\Delta}_q \doteq (m \otimes m) \circ \Delta_q \circ m^{-1} \quad (3.12)$$

we have

$$\widetilde{\Delta}_q(x) = \mathcal{F} \Delta(x) \mathcal{F}^{-1}, \quad \forall x \in U(sl(2))[[h]] \quad (3.13)$$

It is sufficient (and necessary) to verify this equation by substituting for x the image by m of the generators J^0, J^+, J^- .

We remark that there is no loss of generality in restricting ourselves to a specific isomorphism (2.6). Indeed, had we used another isomorphism m' , it turns out from (2.9) that the corresponding \mathcal{F}' would be given by $(M \otimes M) \widetilde{\Delta}_q(M) \mathcal{F}$.

As it is known (cf. [9]) a particular solution up to first order in h is just $\mathcal{F} = 1 + hr$, where

$$r = F \otimes E - E \otimes F \quad (3.14)$$

is the standard classical r-matrix. More generally and up to order two in h we write

$$\mathcal{F} = \mathcal{F}_0 + h\mathcal{F}_1 + h^2\mathcal{F}_2 + o(h^3), \quad (3.15)$$

with \mathcal{F}_i belonging to $U(sl(2)) \otimes U(sl(2))$.

Using (2.8), we obtain the following coupled system of equations to solve by recursion:

$$\begin{aligned} [\mathcal{F}_0, \Delta H] &= 0, \\ [\mathcal{F}_0, \Delta E] &= 0, \\ [\mathcal{F}_0, \Delta F] &= 0. \end{aligned} \quad (3.16)$$

$$\begin{aligned} [\mathcal{F}_1, \Delta H] &= 0, \\ [\mathcal{F}_1, \Delta E] &= (E \otimes H - H \otimes E) \mathcal{F}_0, \\ [\mathcal{F}_1, \Delta F] &= (F \otimes H - H \otimes F) \mathcal{F}_0. \end{aligned} \quad (3.17)$$

$$\begin{aligned}
[\mathcal{F}_2, \Delta H] &= 0 , \\
[\mathcal{F}_2, \Delta E] &= (E \otimes H - H \otimes E)\mathcal{F}_1 - \mathcal{F}_0 \Delta \phi_2^+ \Delta E \\
&\quad + \left(\frac{1}{2} E \otimes H^2 + \frac{1}{2} H^2 \otimes \phi_2^+ E + \phi_2^+ E \otimes 1 + 1 \otimes \phi_2^+ E \right) \mathcal{F}_0 , \\
[\mathcal{F}_2, \Delta F] &= (F \otimes H - H \otimes F)\mathcal{F}_1 - \mathcal{F}_0 \Delta \phi_2^- \Delta F \\
&\quad + \left(\frac{1}{2} F \otimes H^2 + \frac{1}{2} H^2 \otimes \phi_2^- F + \phi_2^- F \otimes 1 + 1 \otimes \phi_2^- F \right) \mathcal{F}_0 . \tag{3.18}
\end{aligned}$$

Besides $\mathcal{F}_0 = 1 \otimes 1$, any arbitrary polynomial f_0 in the variables $(I \otimes 1, 1 \otimes I, \Delta I)$ satisfies equations (3.16). Due to linearity of the equation we can write then:

$$\mathcal{F}_0 = 1 \otimes 1 + f_0 . \tag{3.19}$$

As regards \mathcal{F}_1 , besides the solution $\widetilde{\mathcal{F}}_1 = r$ of the equations (3.17) (with $f_0 = 0$), a solution for the general case is given by

$$\mathcal{F}_1 = \widetilde{\mathcal{F}}_1(1 \otimes 1 + f_0) + f_1 , \tag{3.20}$$

with f_1 being a solution of (3.16).

Similarly for \mathcal{F}_2 : if one finds a particular solution $\widetilde{\mathcal{F}}_2$ of (3.18) (with $f_0 = f_1 = 0$), the most general one is given by

$$\mathcal{F}_2 = \widetilde{\mathcal{F}}_2(1 \otimes 1 + f_0) + \widetilde{\mathcal{F}}_1 f_1 + f_2 , \tag{3.21}$$

with f_2 solution of (3.16).

The possibility of adding *pure kernel* (i.e. satisfying the homogeneous equations (3.16)) terms f_1 and f_2 comes from the fact that the last two equations for \mathcal{F}_1 and \mathcal{F}_2 are linear non homogeneous, whose associated homogeneous ones are the last two equations in (3.16).

Now we proceed to exhibit the aforementioned particular solutions $\widetilde{\mathcal{F}}_i$ of this set of equations. In $U(sl(2)) \otimes U(sl(2))$ we use the basis

$$\{H^{a_1} I^{b_1} E^{c_1} \oplus H^{r_1} I^{s_1} F^{t_1}\} \otimes \{H^{a_2} I^{b_2} E^{c_2} \oplus H^{r_2} I^{s_2} F^{t_2}\} .$$

In order to simplify the notation, for any $x \in U(sl(2))$ we set $x_1 = x \otimes 1$, $x_2 = 1 \otimes x$. From $[\mathcal{F}_i, \Delta H] = 0$, for all i , it is easily seen that any \mathcal{F}_i is of the form $\mathcal{F}_i = a_{il} E_1^l F_2^l + b_{il} F_1^l E_2^l$, where a_{il} and b_{il} are polynomials in H_1, H_2, I_1, I_2 .

We've already mentioned that $\widetilde{\mathcal{F}}_0 = 1$ is a solution for equations (3.16).

Next we pass to the first order term. For simplicity we drop the index $i = 1$ in the following formulae and define

$$\begin{aligned}
\delta_1(a_k) &= a_k(H_1, H_2, I_1, I_2) - a_k(H_1 - 1, H_2, I_1, I_2) , \\
\delta_2(a_k) &= a_k(H_1, H_2, I_1, I_2) - a_k(H_1, H_2 - 1, I_1, I_2) ,
\end{aligned}$$

and similarly for b_k . The equations (3.17) give the following system of coupled partial difference equations for the coefficients a_l and b_l :

$$\begin{aligned}
\delta_1(a_{n-1}) &= -\frac{1}{2}(I_2 + H_2 - H_2^2)\delta_2(a_n) + (nH_2 + \frac{n^2-n}{2})a_n, \\
\delta_2(a_{n-1}) &= -\frac{1}{2}(I_1 - H_1 - H_1^2)\delta_1(a_n(H_1+1, H_2-1)) + (nH_1 - \frac{n^2-n}{2})a_n(H_1, H_2-1), \\
\delta_1(b_{n-1}) &= -\frac{1}{2}(I_2 - H_2 - H_2^2)\delta_2(b_n(H_1-1, H_2+1)) + (nH_2 - \frac{n^2-n}{2})b_n(H_1-1, H_2), \\
\delta_2(b_{n-1}) &= -\frac{1}{2}(I_1 + H_1 - H_1^2)\delta_1(b_n) + (nH_1 + \frac{n^2-n}{2})b_n,
\end{aligned} \tag{3.22}$$

for any $n \geq 2$, whereas for $n = 1$ we have:

$$\begin{aligned}
\delta_1(a_0 + b_0) &= -\frac{1}{2}(I_2 + H_2 - H_2^2)\delta_2(a_1) + H_2a_1 + H_2, \\
\delta_1(a_0 + b_0) &= -\frac{1}{2}(I_2 - H_2 - H_2^2)\delta_2(b_1(H_1-1, H_2+1)) + H_2b_1(H_1-1, H_2) - H_2, \\
\delta_2(a_0 + b_0) &= -\frac{1}{2}(I_1 - H_1 - H_1^2)\delta_1(a_1(H_1+1, H_2-1)) + H_1a_1(H_1, H_2-1) + H_1, \\
\delta_2(a_0 + b_0) &= -\frac{1}{2}(I_1 + H_1 - H_1^2)\delta_1(b_1) + H_1b_1 - H_1.
\end{aligned} \tag{3.23}$$

In order to find a particular solution of this system of equations, one can fix a couple $\{N, K\}$ such that $a_n = b_k = 0$, $\forall n \geq N$ and $\forall k \geq K$, in order to set the maximum degree for the polynomials in $E_1^l F_2^l$ and $E_2^l F_1^l$, and then solve recursively the equations for the lower degree terms by partial finite integration.

By making a *minimal* choice, putting $a_n = b_n = 0$, for any $n \geq 2$, we recover the solution:

$$\widetilde{\mathcal{F}}_1 = r, \tag{3.24}$$

with r given by (3.14). Consistently with what we explained in the previous section, had we decided to fix our cut-off at higher degree terms we would have adjoined to $\widetilde{\mathcal{F}}_1$ some f_1 solution of the *pure kernel* part.

As regards \mathcal{F}_2 , the structure of the equations for a_l and b_l remains unchanged for $n \geq 3$, whereas for $n = \{2, 1\}$ some extra term appear, due to ϕ_2^+ and ϕ_2^- .

We skip the explicit (and lengthy) form of them, and we just give the expression for a particular solution:

$$\begin{aligned}
\widetilde{\mathcal{F}}_2 &= \frac{1}{2}(I \otimes H^2 + H^2 \otimes I) + \frac{1}{3}(E \otimes HF - HE \otimes F + HF \otimes E - F \otimes HE) \\
&\quad + \frac{1}{6}H \otimes H(1 - 3P) - \frac{11}{24}P + \frac{1}{2}((1 + P)^2 - 1 - 2I \otimes I),
\end{aligned} \tag{3.25}$$

where

$$P = 2(E \otimes F + F \otimes E + H \otimes H) \tag{3.26}$$

is the Cartan-Killing metric.

Applying representations of $sl(2)$ we can obtain explicit matrix expressions for $\widetilde{\mathcal{F}}$. It

turns out that our particular solution $\tilde{\mathcal{F}}$, when composed with $\frac{1}{2} \otimes \text{id}$, reproduces the semi-universal solution presented in [8] in terms of 2×2 matrices with coefficients in $U(\mathfrak{sl}(2))[[h]]$ (up to the second order in h). Thus, as a consequence it also coincides with the matrix solutions in the representations $\frac{1}{2} \otimes j$.

We remark that in the literature one may find often other properties of the twisting element \mathcal{F} . For instance, \mathcal{F} may be supposed to satisfy the

i) ‘normalization’ condition

$$(\varepsilon \otimes \text{id})(\mathcal{F}) = (\text{id} \otimes \varepsilon)(\mathcal{F}) = 1 , \quad (3.27)$$

sometimes also expressed as $\mathcal{F}(x, 0) = \mathcal{F}(0, y) = 1$. With the standard definition of counit ε this implies $\mathcal{F}_0 = 1$, i.e. $f_0 = 0$.

ii) unitarity condition $\sigma(\mathcal{F})\mathcal{F} = 1$. In our case \mathcal{F} fulfills this condition in the particular representation $\frac{1}{2} \otimes \frac{1}{2}$, but not in general.

iii) condition $(\mathcal{F} \otimes \text{id})(\Delta \otimes \text{id})\mathcal{F} = (\text{id} \otimes \mathcal{F})(\text{id} \otimes \Delta)\mathcal{F}$. In our case \mathcal{F} does not fulfill it, not even in a representation (except the trivial one). We remark that this condition is a stronger requirement with respect to the coassociativity of the twisted coproduct, which in our case follows directly from the definition.

4 Quasitriangular Hopf algebra level

From the Drinfeld theorem, the quantum universal R-matrix

$$R_q = q^{2J^0 \otimes J^0} \sum_{n=0}^{\infty} q^{-n(n-1)/2} \frac{2^n (1 - q^{-2})^n}{[n]!} (q^{J^0} J^+ \otimes q^{-J^0} J^-)^n , \quad (4.28)$$

and the undeformed universal R-matrix, though not the simple $1 \otimes 1$ but rather,

$$R = q^P , \quad (4.29)$$

with P given by (3.26), should be related by the isomorphism up to a twist. Thus, setting

$$\widetilde{R}_q \doteq (m \otimes m)(R_q) , \quad (4.30)$$

\mathcal{F} is supposed to verify the equation

$$\widetilde{R}_q \mathcal{F} = \sigma(\mathcal{F}) R , \quad (4.31)$$

where σ is the flip operator and $[n]! \doteq [n][n-1] \dots 1$.

We have the following expansions

$$\begin{aligned} R &= 1 + hR^{(1)} + h^2 R^{(2)} + o(h^3) \\ &= 1 + h(2E \otimes F + 2F \otimes E + 2H \otimes H) \\ &\quad + h^2(-H \otimes H - 2E \otimes F - 2F \otimes E + 2E^2 \otimes F^2 + 2F^2 \otimes E^2 \\ &\quad - 2E \otimes HF - 2HF \otimes E + 2F \otimes HE + 2HE \otimes F + 4HE \otimes HF + 4HF \otimes HE \\ &\quad + 3H^2 \otimes H^2 + I \otimes I - I \otimes H^2 - H^2 \otimes I) , \end{aligned} \quad (4.32)$$

$$\begin{aligned}
\widetilde{R}_q &= 1 + hR_q^{(1)} + h^2R_q^{(2)} + o(h^3) \\
&= 1 + h(4E \otimes F + 2H \otimes H) \\
&\quad + h^2(2H^2 \otimes H^2 - 4E \otimes F - 4E \otimes HF + 8E^2 \otimes F^2 + 4HE \otimes F + 8HE \otimes HF) .
\end{aligned} \tag{4.33}$$

At the zero-order in h , choosing $f_0 = 0$, (4.31) is identically satisfied ($1 = 1$).

At the order one we have the following equation:

$$\sigma(f_1) - f_1 = R_q^{(1)} - R^{(1)} - \left(\sigma(\widetilde{\mathcal{F}}_1) - \widetilde{\mathcal{F}}_1 \right) . \tag{4.34}$$

It comes from direct computations that the right-hand-side is zero, which implies that f_1 must be symmetric.

At the second order we obtain

$$\sigma(f_2) - f_2 = \widetilde{\mathcal{F}}_2 - \sigma(\widetilde{\mathcal{F}}_2) - \sigma(\widetilde{\mathcal{F}}_1)R^{(1)} + R_q^{(1)}\widetilde{\mathcal{F}}_1 + R_q^{(2)} - R^{(2)} . \tag{4.35}$$

Again the right-hand-side is zero, and hence also f_2 must be symmetric.

Since, in particular, f_1 and f_2 can be equal to zero, we have that our particular solution $\widetilde{\mathcal{F}}$ satisfies (4.31).

5 Conclusions

In accordance with the theorem of Drinfeld, we have exhibited an isomorphism from $U_h(sl(2))$ to $U(sl(2))[[h]]$ and (up to the second order in h) a class of universal twisting elements $\mathcal{F} \in (U(sl(2)) \otimes U(sl(2)))[[h]]$. Such \mathcal{F} perform a gauge transformation (twist) from the ordinary coproduct and from the universal R-matrix $R = q^P$ in $U(sl(2))[[h]]$ to their quantum counterparts in $U_h(sl(2))$.

We have identified a particular universal element $\widetilde{\mathcal{F}}$ in this class which, after applying the representation $\frac{1}{2}$ to its first leg, coincides with the ‘semi-universal’ solution in [8] (up to the second order in h). Consequently, it also coincides with the known matrix solutions in the representations $\frac{1}{2} \otimes j$.

The computation of the higher order terms, with the help of ‘Mathematica’, is in progress.

References

- [1] V.G. Drinfeld “Quasi-Hopf Algebras and Knizhnik-Zamolodchikov equations” Res. Rep. Phys., 1989, *Springer*
- [2] V.G. Drinfeld “Quasi-Hopf Algebras” *Leningrad Math. J.* 1990 **1** (6) 1419–1457
- [3] M. Bonechi, R. Giachetti, E. Sorace & M. Tarlini *Commun. Math. Phys.* 1995 **169** (243) 627–634
- [4] T.L. Curtright & C.K. Zachos “Deforming maps for quantum algebras” *Phys. Lett. B* 1990 **3** (243) 237–244
- [5] T.L. Curtright “Deformations, Coproducts, and U ” in *Quantum Groups* T.L. Curtright, D.B. Fairlie & C.K. Zachos eds *World Scientific* 1991
- [6] C.K. Zachos “Quantum Deformations” in *Quantum Groups* T.L. Curtright, D.B. Fairlie & C.K. Zachos eds. *World Scientific* 1991
- [7] R.A. Engeldinger “On the Drinfeld-Kohno Equivalence of groups and Quantum Groups” Prep. LMU-TPW 95-13 (q-alg/9509001)
- [8] T.L. Curtright, G.I. Ghandour, C.K. Zachos “Quantum algebra deforming maps, Clebsch-Gordan coefficients, coproducts, R and U matrices” *J. Math. Phys.* 1991 **32** (3) 676–688
- [9] L.A. Takhtajan “Lectures on Quantum Groups” in *Introduction to quantum group and integrable massive models of quantum field theory* M. Ge & B. Zhao eds. *World Scientific* 1989
- [10] L. Dąbrowski “Drinfeld twisting and nonstandard quantum groups” in Proc. 10th Naz.Conv.Gen.Rel., Bardonecchia 1992; 661-665, *World Scientific*
- [11] C. Kassel *Quantum Groups* Springer-Verlag 1995